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A Condition for the Existence of a Weakly Closed TI-Set

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The purpose of this paper is to prove the following:

THEOREM. *Let G be a finite group with* $O_2(G) = 1$ *and M a maximal 2-local subgroup of G satisfying* $F^*(M) = O_2(M) = Q$. *Suppose a maximal elementary Abelian normal subgroup V of M is a TI-set in G. Then one of the following holds*:

(1) *V is weakly closed in M (and so* $\langle V^G \rangle$ *known by* [11] *or*

(2) *for all subgroups H of G, satisfying* $O_2(H) \neq 1$ *and* $P = C_Q(V) \leqslant H$, $V \leqslant O_2(H)$ *follows.*

Assume that hypothesis of the above theorem is satisfied. Then Theorem 2 of [14] shows that one of the following holds:

(a) Q is of symplectic type and $|V| = 2$.

(b) $|V| > 2$, $V = Z(Q) = \Phi(Q) = Q'$, and $Q = \langle V^g \mid V^g \leqslant Q \rangle$.

(c) $|V| > 2$ and V is weakly closed in Q.

Further, in (b) and (c) V is a maximal Abelian normal subgroup of M.

Simple groups satisfying (a) are completely classified. In case (b) Stroth has shown that one obtains for the structure of M, apart from the sporadic group cases, possibilities similar to those in the extraspecial case in [13]. Steve Smith is working on the final classification of these groups. Besides being used by Aschbacher in his work on strongly p-embedded subgroups, the above theorem is essential to the treatment of case (c) in [15]. Namely, if V is weakly closed in Q but not in M, we show in Section 3 of [15] that $\bar{P}$ is a weakly closed TI-set in $\bar{H}$ for each H satisfying $P \leqslant H \nleqslant M$ and $O_2(H) \neq 1$, where $\bar{H} = H/C_H(W)$ and $W = \langle V^H \rangle$. By [11] this determines the structure of $\langle \bar{P}^{\bar{H}} \rangle$, giving us detailed information about all 2-local subgroups of G which contain P but are not contained in M. This is the main motivation for proving the above theorem.

It should be mentioned that Aschbacher proved in [3, (9.3)] similar theorems in his special situation, thereby giving me some ideas on how to treat the problem.

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1. Preliminary Results

(1.1) Lemma. *Suppose G is a finite group with $O_2(G) = 1$, M a maximal 2-local subgroup of G such that $F^*(M) = O_2(M) = Q$. Suppose some maximal elementary Abelian normal subgroup V of M os a TI-set in G and $|V| > 2$. Then*

(1) $\langle V^G \rangle \simeq L_3(4)$ *and V is a root-subgroup of* $\langle V^G \rangle$ *or*

(2) *if $P = C_Q(V)$ then $C(P) = V$ and $\Phi(P) \leqslant V$.*

Proof. Theorem 1 of [14] implies that either (1) holds or V is a maximal Abelian normal subgroup of M. But then (2) is [14, (2.9)].

(1.2) Lemma. *Let $X \simeq U_n(q)$ or* $\mathrm{Sp}(2n, q)$, $q = 2^m$, $n \geqslant 3$ *resp.* $n \geqslant 2$, *V a root subgroup of transvections of X, and $Q = O_2(C_X(V))$. Then*

(1) $X = \langle Q, V^g \rangle = \langle Q, a \rangle$ *for each* V^g, $g \in X$, *such that* $V^g \leqslant C_X(V)$ *and* $a \in (V^g)^\#$.

(2) $C_X(V) = Q(C_X(V) \cap C_X(V^g)), Q \cap (C_X(V) \cap C_X(V^g)) = 1$, *where* $V^g \nleqslant C(V)$.

(3) *If $A \leqslant \mathrm{Aut}(X)$ is a 2-group centralizing V and satisfying $[C_X(V), A] \leqslant Q$, then $QA = Q \times C_{QA}(X)$.*

Proof. By [1] Q acts regularly on $\mathfrak{M} = \{V^g : V^g \nleqslant C(V)\}$. Hence $X_1 = \langle Q, V^g \rangle$ contains with each conjugate V^h all elements of V^X which do not centralize V^h. Now, if V corresponds to the point P in the natural representation of X, then $\mathfrak{M}$ corresponds to the set of all points which are nonperpendicular to P. This shows $X_1 = \langle V^X \rangle = X$. $X = \langle Q, a \rangle$ follows from $\langle V, V^g \rangle = \langle V, a \rangle \simeq L_2(q)$. This proves (1).

By (1) we have $C_X(V) = Q \cdot R$, where $R = C(V) \cap N(V^g)$ and $Q \cap R = 1$. The structure of the automorphism group of $L_2(q)$ implies $R = C(V) \cap C(V^g)$ proving (2).

If $X \simeq U_3(q)$ then(3) is [7, (3.3)]. So assume $n \geqslant 4$ if $X \simeq U_n(q)$. Then in any case $C_X(V)$ acts irreducibly on Q/V. Now $QA < C_X(V)A$ and so by the above $QA = Q \times B$ and $B = C_{QA}(V^g)$, where $V^g \leqslant C_X(V)$. Hence $R \leqslant N_X(B)$ and so $[R, B] \leqslant Q \cap B = 1$. Since R is irreducible on Q/V and $C_{Q/V}(B) \neq 1$ this shows $[Q, B] \leqslant V$, whence $[C_X(V), B] \leqslant V$ by (2). Since $C_X(V) = C_X(V)'$ this shows $[C_X(V), B] = 1$ and so $[X, B] = 1$ by (1). This proves (3).

(1.3) Lemma. *Let $X \simeq \mathrm{Sp}(2n, q)$ or $U_n(q)$, $q = 2^m$, and let V be a root subgroup of transvections of X. Then*

(1) *If $X \simeq U_n(q)$ then X is generated by n conjugates of V.*

(2) *If $X \simeq \mathrm{Sp}(2n, q)$, $q > 2$, then X is generated by $2n$ conjugates of V.*

(3) *If $X \simeq \mathrm{Sp}(2n, 2)$ then X is generated by $2n + 1$ conjugates of V.*

Proof. Condition (1) may be easily proved by induction on n. To prove (2) let $Q = O_2(C_X(V))$. Then [1] implies that there is a complement R of Q in $C_X(V)$, which is generated by conjugates of V and which centralizes some $V^g \leqslant C_X(V)$. Further $R \simeq \mathrm{Sp}(2n-2, q)$ and Q is elementary Abelian and an indecomposable F_2R-module, which is mod V the natural module. By induction R is generated by $2n-2$ conjugates of V. Hence $C_X(V)$ is generated by $2n-1$ conjugates of V. But then (1.2) implies that X is generated by $2n$ conjugates of V.

Condition (3) was shown in [13, (2.3)].

(The reason why one needs one more element of V^X in case $q=2$ and $X \simeq \mathrm{Sp}(2n, q)$ is that if $X \simeq \mathrm{Sp}(4, 2)$ then $C_X(V)$ is not indecomposable on Q.)

(1.4) LEMMA. *Suppose $\bar{X} = X/N \simeq \mathrm{Sp}(2n, q)$, $q = 2^m$, N is elementary Abelian, and $X = \langle V^X \rangle$, where V is an elementary Abelian 2-subgroup which is a TI-set in X. Suppose further that $V \cap N \neq 1$ and $C_N(X) = 1$. Then*

(1) *$\bar{V}$ is a root subgroup of transvections of $\bar{X}$, and*

(2) *N is the direct sum of natural $F_2\bar{X}$-modules.*

Proof. By (7.2), (7.3) of [4] we have that $\bar{V}$ is a TI-set of $\bar{X}$, $[V, V^h] = 1$ if $\langle \bar{V}, \bar{V}^h \rangle$ is a 2-group, and $\bar{D}$ is a set of root involutions of $\bar{X}$ where $D = \{v \in V^h \mid h \in X\}$. The TI-set property of $\bar{V}$ together with the root-involution property of $\bar{D}$ implies by [10] that $\bar{V}$ is a subgroup of a root subgroup $\bar{A}$ of $\bar{X}$ corresponding to a short or long root of the root system of $\bar{X}$. Now $[V, V^h] = 1$ if $\langle \bar{V}, \bar{V}^h \rangle$ is a 2-group implies that $\bar{A}$ is a root group of transvections.

Suppose $\bar{V} < \bar{A}$. Let $\bar{V} \neq \bar{V}^h \leqslant \bar{A}$. By what we have shown in (1.3) $C_{\bar{X}}(\bar{V}) = C_{\bar{X}}(\bar{V}^h) = C_{\bar{X}}(\bar{A}) = \langle \bar{V}^g \mid \bar{V}^g \leqslant C_{\bar{X}}(\bar{V}) \rangle$. Hence $[V \cap N, C_{\bar{X}}(\bar{V})] = 1 = [V^h \cap N, C_{\bar{X}}(\bar{V})]$ by the above. Let $1 \neq \bar{a} \in \bar{V}^g \not\leqslant C_{\bar{X}}(\bar{V})$. Then by (1.2), $\bar{X} = \langle C_{\bar{X}}(\bar{V}), \bar{a} \rangle$ and $[N, \bar{a}] \leqslant V^g \cap N$. Thus if $V \cap N \neq V^h \cap N$ then $\bar{a}$ centralizes some element of $(V \cap N)(V^h \cap N)^{\#}$ and thus $C_N(\bar{X}) \neq 1$, contradicting the assumption. This proves (1).

Now $N_{\bar{X}}(\bar{V}) = C_{\bar{X}}(\bar{V})\langle \bar{h} \rangle$, $o(\bar{h}) = q-1$, and $\langle \bar{h} \rangle$ acts regularly on $\bar{V}^{\#}$. Since $\langle \bar{V}, \bar{V}^g \rangle \simeq L_2(q)$ for some $g \in X$, [12, (2.4)] implies that $\langle \bar{h} \rangle$ acts regularly on $(V \cap N)^{\#}$. Hence [14, (6.6)] implies that $V \cap N = \bigoplus_{j=1}^{k} V_j$, $|V_j| = q$, and each V_j is a faithful irreducible $F_2\langle \bar{h} \rangle$-module. Since $C_{\bar{X}}(\bar{V})$ centralizes by the above $V \cap N$ the V_i are $N_{\bar{X}}(\bar{V})$ invariant.

Let $\bar{V}^1, \ldots, \bar{V}^l$ (by (1.3)) be a system of conjugates of $\bar{V}$, which generate $\bar{X}$, where $\bar{V}^1 = \bar{V}$ and $l = 2n$ resp. $2n-1$ in case $q = 2$. Let $\bar{x}_1 = 1$ and $\bar{x}_2, \ldots, \bar{x}_l \in \bar{X}$ such that $(\bar{V}^1)^{\bar{x}_i} = \bar{V}^i$. Then $V_j^{\bar{x}_i}$ is $N_{\bar{X}}(\bar{V}^i)$ invariant for each $j = 1, \ldots, k$, $i = 1, \ldots, l$. If $[\bar{V}^i, \bar{V}^s] = 1$ then by the above $V_j^{\bar{x}_i} \dotplus V_j^{\bar{x}_s}$ is centralized by $\langle \bar{V}^i, \bar{V}^s \rangle$. So assume that $\langle \bar{V}^i, \bar{V}^s \rangle \simeq L_2(q)$. Since $V_j^{\bar{x}}$ is $N_X(V^i)$ invariant it follows, arguing as in [11, (3.11)], that $V_j^{\bar{x}_i} \dotplus V_j^{\bar{x}_s}$ is $\langle \bar{V}^i, \bar{V}^s \rangle$ invariant. Hence $[\bar{V}^i, V_j^{\bar{x}_s}] \leqslant V_j^{\bar{x}_i}$ for $j = 1, \ldots, k$.

Let $W_j = \sum_{i=1}^{l} V_j^{x_i}$, $j = 1,..., k$. Then we have shown that W_j is $\langle \bar{V}^{x_1},..., \bar{V}^{x_l}\rangle = \bar{X}$ invariant. If $q > 2$ then $|W_j| \leqslant q^{2n} = q^l$, whence immediately $|W_j| = q^{2n}$ and V_j^X is a partition of W_j, since $|\bar{X} : N_{\bar{X}}(\bar{V})| = (q^{2n} - 1)/(q - 1)$. But then, arguing as in [11, (2.9)] W_j is the natural $F_2\bar{X}$-module.

Assume next that $q = 2$. Then $|W_j| \leqslant 2^{2n+1}$ and $|[W_j, \bar{V}]| = 2$, since $[W_j, \bar{V}] = V_j$. Hence $\bar{V}$ induces a transvection on W_j. Since $[W_j, \bar{X}] = W_j$ it follows easily that either W_j is the natural $F_2\bar{X}$-module or W_j is the extension of a trivial by the natural module. But the second case contradicts $C_W(X) = 1$.

So in any case W_j is the natural $F_2\bar{X}$-module for $j = 1,..., k$. Claim $N = \bigoplus_{j=1}^{k} W_j$. Let $W^s = \sum_{j=1}^{s} W_j$ and show $W^s = \bigoplus_{j=1}^{s} W_j$ by induction on s. Suppose this holds for s. Then $C_{W^s}(C_{\bar{X}}(\bar{V})) = V \cap W^s = \bigoplus_{j=1}^{s} V_j$ but $C_{W_{s+1}}(C_{\bar{X}}(\bar{V})) = V_{s+1}$. Hence $W^s \cap W_{s+1} = 0$ and thus $W^{s+1} = W^s \oplus W_{s+1} = \bigoplus_{j=1}^{s+1} W_j$. This proves (2).

(1.5) LEMMA. *Let $X \simeq \mathrm{Sp}(2n, q)$, $q = 2^m$, $n \geqslant 2$, and let V be an indecomposable F_2X-module which contains the natural F_2X-module W such that V/W is a trivial module. Then*

(1) $|V/W| \leqslant q$.

(2) *Let $A < X$ be a root group of transvections and $Q = O_2(C_X(A))$. Then $V = W + U$, where $[Q, U] \leqslant [W, A]$.*

Proof. If $q = 2$ then (1) is [13, (2.3)]. So assume that $q > 2$. Then (1.3) implies that X is generated by $2n$ conjugates of A. Let $a \in A^\#$ and $B \sim A$ in X such that $\langle A, B\rangle \simeq L_2(q)$. Then there exist $b, c \in B^\#$ such that $\langle A, B\rangle = \langle a, b, c\rangle$. This implies that X is generated by $2n - 1$ conjugates of a, say $a_1 ,..., a_k$, $k = 2n - 1$.

Since $C_X(a)$ normalizes $[V, a] \leqslant W$ the action of X on its natural module implies $|[V, a]| = q$. Hence $|V : C_V(a)| = q$ and so $|V : C_V(X)| \leqslant q^{2n-1}$, since $\bigcap_{i=1}^{k} C_V(a_i) \leqslant C_V(X)$. But the indecomposability of V implies that $C_V(X) = 0$ and thus $|V| \leqslant q^{2n-1}$ and $|V/W| \leqslant q$. This proves (1).

To prove (2) let $G = V \cdot X$, $C_X(A) = Q \cdot R$, $R \simeq \mathrm{Sp}(2n - 2, q)$, $C = [W, A]$, and $W_0 = C_W(A)$. Then $|C| = |W : W_0| = q$. Furthermore Q is elementary and Q/A and W_0/C are natural F_2R-modules.

Since $|[V, a]| = q$ it follows that $[V, A] = C$. Hence $[V, QR] \leqslant W_0$ since QR is generated by conjugates of A. As shown $AC \lhd VQR$. Let $\overline{VQ} = VQ/AC$. Since $[Q, W_0] \leqslant C$ it follows that $\bar{Q}\bar{W}_0$ is elementary Abelian and $\bar{W}_0$ and $\bar{Q}\bar{W}_0/\bar{W}_0$ are both natural F_2R-modules. Since R centralizes $\bar{V}/\bar{W}_0$ and since $[\bar{Q}\bar{W}_0, \bar{V}] \leqslant \bar{W}_0$ each $\bar{x} \in \bar{V}$ induces by the commutator map an F_2R-homomorphism from $\bar{Q}\bar{W}_0/\bar{W}_0$ in $\bar{W}_0$.

Now easily $\mathrm{Hom}_{F_2R}(\bar{Q}\bar{W}_0/\bar{W}_0, \bar{W}_0) \simeq F_q$, since both are natural F_2R-modules. Hence $|\bar{V} : C_{\bar{V}}(\bar{Q}\bar{W}_0)| \leqslant q$. By the action of X on its natural module $[W, Q] \leqslant W_0$. Hence $C_{\bar{W}}(\bar{Q}\bar{W}_0) = \bar{W}_0$ and thus $\bar{V} = \bar{W} \cdot C_{\bar{V}}(\bar{Q}\bar{W}_0)$. Let $U \leqslant V$ such that $\bar{U} = C_{\bar{V}}(\bar{Q}\bar{W}_0)$. Then $V = W + U$ and $[Q, U] \leqslant AC \cap W = C$. This proves (2).

2. The Structure of H

Assume in this section that the hypothesis of the theorem is satisfied but the conclusion is false. Let $\mathfrak{H} = \{H \leqslant G \mid P \leqslant H \text{ and } O_2(H) \neq 1\}$. Choose $H \in \mathfrak{H}$ minimal subject to $V \nleqslant O_2(H)$ and fix H for the rest of this section. Let $\Omega = V^H$, $X = \langle \Omega \rangle$, and $W = \langle U \cap O_2(H) \mid U \in \Omega \rangle$. Let $D = \{d \in U^{\#} \mid U \in \Omega\}$ and $\bar{X} = X/W$.

(2.1) *W is elementary Abelian. $F^*(X) = O_2(X)$ and $\overline{F^*(X)} \leqslant Z(\bar{X})$. Furthermore $\bar{D}$ is a set of root involutions of $\bar{X}$.*

Proof. Since, by (1.1), $C(P) = V$, $Z(O_2(H)) \cap V \neq 1$. But then by [12, (2.4)], $[V \cap O_2(H), U \cap O_2(H)] = 1$ for $V, U \in \Omega$. Hence W is elementary. By [12, (2.5)] $\bar{D}$ is a set of root involutions of $\bar{X}$.

Let $h \in F^*(X)$ be of odd order. Then $[V, h] = 1$, whence $h \in O(Z(X))$. But then $[P, h] = 1$ and thus $h = 1$. Hence $F^*(X) = O_2(X)$. Since $[V, O_2(X)] \leqslant V \cap W$, (2.1) follows.

(2.2) *The following holds:*

1. *If $h \in H$ and $\langle \bar{V}, \bar{V}^h \rangle$ is a 2-group, then $[V, V^h] = 1$.*
2. *$\bar{V}$ is a TI-set in $\bar{H}$ and $N_{\bar{H}}(\bar{V}) = N_H(V)/W$.*
3. *$C_W(X) = 1$ and $W = [W, X]$.*
4. *$\bar{D}$ is a class of root involutions of $\bar{X}$.*

Proof. Both (1) and (2) are in [4, (7.3)]. By [12, (2.4)] and (1), $V \cap W \leqslant [W, X]$. Suppose $C = C_W(X) \neq 1$. Since $P \leqslant N(C)$, $C \cap V \neq 1$. But this contradicts V a TI-set in H.

Suppose (4) is false. Let $\bar{D}_i$, $1 \leqslant i \leqslant r$, be the orbits of $\bar{X}$ on $\bar{D}$. Then by [10, (4.15), (4.16)], $\bar{X}_i = \langle \bar{D}_i \rangle$ is transitive on $\bar{D}_i$ and $[\bar{X}_i, \bar{X}_j] = 1$ for $i \neq j$. Let $d \in V \setminus (V \cap W)$ and assume without loss that $\bar{d} \in \bar{D}_1$. Then $\bar{d} \in \bar{D}_1 \cap \bar{D}_1{}^h$ for each $h \in P$. Hence $\bar{D}_1 = \bar{D}_1{}^h$ and $P \leqslant N(\bar{D}_1)$. But then $X_1P \in \mathfrak{H}$ and $V \nleqslant O_2(X_1P)$. Hence $H = X_1P$ by the minimality of H and $X = X_1$.

(2.3) *Let $q = |\bar{V}|$. Then one of the following holds:*

1. *$q = 2$ and $\bar{X} = O(\bar{X})\bar{V}$.*
2. *$\bar{X} \simeq L_2(q)$, $Sz(q)$ or $\bar{X}/Z(\bar{X}) \simeq U_3(q)$, $q > 2$.*
3. *$\bar{X}/Z(\bar{X}) \simeq \mathrm{Sp}(2n, q)$, $n \geqslant 2$.*
4. *$\bar{X}/Z(\bar{X}) \simeq U_n(q)$, $n \geqslant 4$.*

Further, $Z(\bar{X}) \leqslant \bar{X}'$.

Proof. Suppose false. First assume that $\bar{D}$ is a nondegenerate class of root involutions. Then by (2.2) and [10, Part I and (11.2.9)] either $\bar{V}$ is contained in a root subgroup of $\bar{X}$ or $\bar{X}/Z(\bar{X}) \simeq L_n(2^m)$ and $\bar{V}$ corresponds to the set of transvections corresponding to a fixed point or $\bar{X}/Z(\bar{X}) \simeq A_6$.

In the first case there exists a $U \in \Omega$, such that $\langle \bar{V}, \bar{U} \rangle$ is a non-Abelian 2-group, contradicting (2.2)(1). In the second case, since $\bar{P} \leqslant C(\bar{V})$, the structure of the automorphism group of $L_n(2^m)$ implies that $\bar{H} = \bar{X}\bar{P} = \bar{X} \times C_{\bar{P}}(\bar{X})$, $\bar{P} \cap \bar{X} = \bar{V}$. Now, since $\langle V, V^h \rangle / O_2 \langle V, V^h \rangle \simeq L_2(2^m)$ for each $h \in H$, the minimality of H implies that $n = 2$ and $\bar{D}$ is degenerate. In the third case, since $\bar{P} \leqslant O_2(N_{\bar{H}}(\bar{V}))$, there exists an $h \in H$ such that $\langle \bar{P}, \bar{V}^h \rangle \simeq C_2 \times \Sigma_4$. But by the minimality of H we had $\bar{H} = \langle \bar{P}, \bar{V}^h \rangle$ for each $V^h \leqslant M$, a contradiction.

Hence $\bar{D}$ is a class of odd transpositions of $\bar{X}$. Let $\bar{N} \lhd \bar{H}$, $\bar{N} \leqslant \bar{X}$. If $V \leqslant O_2(NP)$, then $[\bar{V}, \bar{N}] \leqslant O_2(\bar{N}) \leqslant O_2(\bar{X})$. Hence $N \leqslant N(O_2(X)V)$. But then $N \leqslant N(V)$, since by (2.2)(4) $\bar{V}$ is weakly closed in $\bar{V} \times O_2(\bar{X})$. This shows $[N, V] \leqslant V \cap O_2(X) \leqslant W$ and so $\bar{N} \leqslant Z(\bar{X})$.

Thus the minimality of H implies $H = NP$ for each characteristic subgroup $\bar{N}$ of $\bar{X}$, which is not contained in $Z(\bar{X})$. Especially, if $O(\bar{X}) \leqslant Z(\bar{X})$, then $q = 2$ and $\bar{X} = O(\bar{X})\bar{V}$, whence (1) holds.

So we may assume $O(\bar{X}) \leqslant Z(\bar{X})$. Hence by (2.1), $S(\bar{X}) = Z(\bar{X})$. If $\bar{X}$ is solvable, then $O(\bar{X}) \nleqslant Z(\bar{X})$ and so (1) holds. Hence $\bar{X}^{(\infty)} \nleqslant Z(\bar{X})$ and so by the above $\bar{X} = \bar{X}^{(\infty)}\bar{V}$. Especially $\bar{X}' = \bar{X}''$. Hence $\bar{X}/Z(\bar{X})$ satisfies the hypothesis of the main theorem of [1]. By the above, certainly $\bar{X}/Z(\bar{X}) \not\simeq L_2(q) \wr \Sigma_n$, $q = 2^m$, $n \geqslant 3$. Hence by [1]:

(a) $\bar{X}/Z(\bar{X}) \simeq L_2(2^m)$, $Sz(2^m)$, $U_n(2^m)$, or $Sp(2n, 2^m)$;

(b) $\bar{X}/Z(\bar{X}) \simeq O_n^{\epsilon}(q)$, $q = 2^m$, or $q = 3, 5$;

(c) $\bar{X}/Z(\bar{X}) \simeq \Sigma_n$; or

(d) $\bar{X}/Z(\bar{X}) \simeq M(22)$, $M(23)$, or $M(24)$.

Suppose (b), (c), or (d) holds. Let $\tilde{H} = \bar{H}/Z(\bar{X})$. By the structure of these groups $|\bar{V}| = |\tilde{V}| = 2$ and $\tilde{V} = O_2(C_{\tilde{X}}(\tilde{V}))$, since $\bar{V} < C_{\bar{X}}(\bar{v})$ for $\bar{v} \in \bar{V}^{\#}$. Hence $[C_{\tilde{X}}(\tilde{V}), P] \leqslant \tilde{P} \cap C_{\tilde{X}}(\tilde{V}) \leqslant \tilde{V}$. Since $\bar{e}\bar{d} \notin \bar{D}$ for $\bar{e}, \bar{d} \in D$ such that $[\bar{e}, \bar{d}] = 1$, this implies that $[\tilde{C}, \tilde{P}] = 1$ for $\tilde{C} = \langle C_{\tilde{X}}(\tilde{V}) \cap \tilde{D} \rangle$. Since $\tilde{H} = \langle \tilde{P}, \tilde{e} \rangle$ for each $\tilde{e} \in \tilde{D} \setminus \tilde{C}$ by the minimality of H, this implies $C_{\tilde{C}}(\tilde{e}) = 1$ for all such $\tilde{e}$. This is obviously not the case in these groups.

So (a) holds. Next we show $2^m = |\bar{V}| = q$. If $m = 1$, then $|\bar{V}| = 2$ since $\bar{V}$ is a TI-set in $\bar{X}$. So assume $m > 1$. Then $Z(\bar{X}) \leqslant O(\bar{X})$ by [8]. Further, by the structure of the groups listed in (a) and the TI-set property of $\bar{V}$ one may identify $\bar{V}$ with a subgroup of a root subgroup of transvections $\bar{A}$ of $\bar{X}$. If $\bar{V} = \bar{A}$, then $|\bar{V}| = |\bar{A}| = 2^m$. So $\bar{V} < \bar{A}$ and there exists a $\bar{V} \neq \bar{V}^h \leqslant \bar{A}$.

Suppose first that $\bar{X}/Z(\bar{X}) \not\simeq U_3(2^m)$. Let $\bar{U} = C_{\bar{X}}(\bar{V})$. Then $\bar{U} = \langle \bar{V}^g \mid V^g \in \Omega, V^g \leqslant U \rangle$, and $\bar{U} = C_{\bar{X}}(\bar{V}^h)$. Hence by (2.2)(1) $F = (V \cap W)(V^h \cap W) \leqslant$

$C_W(U)$. Further, by the structure of these groups (1.2) there exists a $\bar{V}^r$, $r \in X$, and $\bar{a} \in \bar{V}^r$ such that $\bar{X} = \langle \bar{U}, \bar{a} \rangle$. Hence by (2.2)(3) $W = [W, \bar{U}][W, \bar{a}] = [W, U](V^r \cap W)$. Now $|F| > |V^r \cap W|$, but $[F, \bar{a}] \leqslant V^r \cap W$. Hence $C_F(\bar{a}) \neq 1$, contradicting $C_F(\bar{a}) \leqslant C_W(X) = 1$.

Suppose finally that $\bar{X}/Z(\bar{X}) \simeq U_3(2^m)$. Let $\bar{T} \in \mathrm{Syl}_2(\bar{H})$ containing $\bar{P}$ and $\bar{S} = \bar{T} \cap X$. Then $\bar{S} \in \mathrm{Syl}_2(\bar{X})$ and $Z(\bar{S}) = \Omega_1(\bar{S}) = \bar{Z}$. Suppose $\bar{P} \cap \bar{S} \nleqslant \bar{Z}$. Then $\bar{Z} = [\bar{S}, \bar{P} \cap \bar{S}] \leqslant \bar{P}$. Hence $\bar{P} \cap \bar{S} > \bar{Z}$. But then there exist some $\bar{a} \in \bar{V}^g, g \in X$, such that $\bar{X} = \langle \bar{P} \cap \bar{S}, \bar{a} \rangle$. Now, as above, $C_W(\bar{P} \cap \bar{S}) = V \cap W$, since $C_W(\bar{P} \cap \bar{S}) \cap C_W(\bar{a}) \leqslant C_W(\bar{X}) = 1$. Hence $C_W(\bar{S}) \leqslant V \cap W$ and thus $N_{\bar{X}}(\bar{S}) \leqslant N_{\bar{X}}(\bar{V})$. But then $\bar{V} = \bar{Z}$ and $q = |\bar{V}| = 2^m$.

So we may assume $\bar{P} \cap \bar{S} \leqslant \bar{Z}$. Hence $[\bar{P}, \bar{S}] \leqslant \bar{P} \cap \bar{S} \leqslant \bar{Z}$ and so $\bar{P}$ induces by [7, (3.3)] inner automorphisms according to elements of $\bar{S}$ on $\bar{X}$. Hence $\bar{S}\bar{P} = \bar{S} \times \bar{C}$, $\bar{C} = C_{\bar{S}\bar{P}}(\bar{X})$, and $\bar{H} = \bar{X}\bar{P} = \bar{X} \times \bar{C}$. This implies that $N_{\bar{X}}(\bar{S}) \leqslant N_{\bar{X}}(\bar{S}\bar{P})$. But, since $C_W(\bar{P}) = V \cap W$, $\bar{P}$ is weakly closed in each 2-subgroup of $\bar{H}$ containing $\bar{P}$. Hence $N_{\bar{X}}(\bar{S}) \leqslant N_{\bar{X}}(\bar{P}) \leqslant N_{\bar{X}}(\bar{V})$ and thus $\bar{V} = \bar{Z}$ and $2^m = |\bar{V}| = q$.

(2.4) *Suppose* (2), (3) *or* (4) *of* (2.3) *occurs. Then*

(a) $\bar{V}$ *is a root subgroup of* $\bar{X}$.

(b) $\bar{P} \cap \bar{X} = O_2(N_{\bar{X}}(\bar{V}))$.

Proof. Condition (a) was already shown in the proof of (2.3). Suppose (b) is false. Since $N_{\bar{X}}(\bar{V})$ acts in these groups irreducibly on $O_2(N_{\bar{X}}(\bar{V}))/\bar{V}$ we have $\bar{P} \cap \bar{X} = \bar{V}$. Hence $[N_{\bar{X}}(\bar{V}), \bar{P}] \leqslant \bar{P} \cap \bar{X} = \bar{V}$. If (2) holds, then by [7, (3.3)], $\bar{P}$ induces inner automorphisms according to elements of $\bar{V}$ on $\bar{X}$. Hence $\bar{P} = \bar{V} \times C_{\bar{P}}(\bar{X})$.

If (3) or (4) holds then $C_{\bar{X}}(\bar{V}) = C_{\bar{X}}(\bar{V})'$, whence $[C_{\bar{X}}(\bar{V}), \bar{P}] = 1$. But then (1.2) implies again that $\bar{P} = \bar{V} \times C_{\bar{P}}(\bar{X})$.

So in any case $P = VC_P(W) = V \cdot C_P(\bar{X})$, since $C_P(\bar{X}) \leqslant C(V)$ and so $C_P(\bar{X}) \leqslant C_P(W)$. Now the three-subgroup lemma implies $C_P(\bar{X})' \leqslant C_V(X) = 1$. Hence P is Abelian and thus $P = V$ by (1.1). But then V is a weakly closed TI-set, contradicting the assumption of this section.

(2.5) *Suppose case* (2), (3), *or* (4) *of* (2.3) *holds. Then*

(a) $X = \langle P \cap X, V^g \rangle, g \in X$.

(b) $W = (P \cap W) \times (V^g \cap W)$.

(c) $H = X \cdot C_P(\bar{X})$ *and* $WC_P(\bar{X})$ *is elementary Abelian.*

Proof. Condition (a) follows from (2.4) and (1.2). Hence by (2.2), $W = [P \cap X, W][V^g, W] \leqslant (P \cap W)(V^g \cap W) \leqslant W$. If $(P \cap W) \cap V^g \neq 1$, then $[V, V^g] = 1$ contradicting (a). Thus (b) holds.

To prove (c) note that if case (2) holds [7, (3.3)] implies that $\bar{P}$ induces inner automorphisms according to $\bar{P} \cap \bar{X}$ on $\bar{X}$. Hence $\bar{P} = (\bar{P} \cap \bar{X})\, C_{\bar{P}}(\bar{X})$. If (3) or (4) holds, then (1.2) implies the same statement.

So $P = (P \cap X)\, C_P(\bar{X})$ and, as in (2.4), $C_P(\bar{X}) \leqslant C_P(W)$ and is Abelian. Let $x \in C_P(W)$ be of order 4. Then $C_{V^g}(x) = V^g \cap W = C_{V^g}(x^2)$, since $x^2 \in V \cap W$, contradicting [14, (1.1)]. Hence $W \cdot C_P(\bar{X})$ is elementary.

3. Proof of the Theorem

In this section we continue with the hypothesis and notation of Section 2. We will show that each of the cases of (2.3) leads to a contradiction.

(3.1) *Case* (1) *of* (2.3) *cannot occur.*

Proof. Suppose false. Since $\langle V, V^g\rangle$ is not a 2-group for each $V^g \neq V$, $g \in H$, the minimality of H implies $H = \langle P, V^g\rangle$. Hence as in (2.5), $W = (P \cap W) \times (V^g \cap W) = (P^g \cap W) \times (V \cap W)$. Since each element in $P \cap P^g \cap W$ is centralized by $\langle V, V^g\rangle$, [11, (2.4)] implies that

$$(V \cap W)(V^g \cap W) \cap (P \cap P^g \cap W) = 1.$$

Hence

$$W = (V \cap W)(V^g \cap W)(P \cap P^g \cap W). \tag{*}$$

Let $\tilde{H} = H/C_H(W)$. Suppose $\tilde{x} \in \tilde{P} \cap \tilde{P}^g$. Since $[\tilde{x}, P \cap P^g \cap W] \leqslant (V \cap W) \cap (V^g \cap W) = 1$, (*) implies that $\tilde{x} = 1$. Hence $\tilde{P}$ is a TI-set in $\tilde{H}$, since (*) holds for each $g \in H \setminus N(V)$. Further, since $C_W(\tilde{P}) = V \cap W$, $\tilde{P}$ is weakly closed.

Now, since $O(\tilde{H}) \nleqslant Z(\tilde{H})$ and $\Phi(P) \leqslant V$, $\tilde{P} \simeq C_2$, C_4, or Q_8. In any case $\Omega_1(\tilde{P}) = \tilde{V}$, whence $\Omega_1(P) \leqslant (P \cap C(W))V$. But then $V(P \cap W) \leqslant Z(\Omega_1(P))$ and thus $P \cap W = V \cap W$, since V is a maximal Abelian characteristic subgroup of P.

So by (*), $W = (V \cap W)(V^g \cap W)$. Suppose $\tilde{x} \in \tilde{P}$ is of order 4. Then $C_W(\tilde{x}) = V \cap W = C_W(\tilde{x}^2)$, since $\tilde{x}^2 \in \tilde{V}$, again a contradiction to [14, (1.1)]. Thus $\tilde{V} = \tilde{P}$ and $P = VC_P(W)$. As in (2.5), $C_P(W)$ is elementary. Let $I = WC_P(W)$. Then $C_I(P) = C_I(V) = C_I(v)$ for $v \in V \setminus (V \cap W)$. So, if $C_P(W) > P \cap W$, then $C_I(P) > V \cap W$, since $[I, v] \leqslant V \cap W$. But this contradicts $C(P) = V$. So $P = V(P \cap W) = V$ and V is weakly closed, contradicting the assumption of this section.

(3.2) *Case* (2) *of* (2.3) *cannot occur.*

Proof. Suppose false. Then by (2.5)(c), $\Omega_1(P) = VC_P(\bar{X})$ is elementary Abelian. Hence $\Omega_1(P) = V$ and $V \cap W = P \cap W = C_P(\bar{X})$. But then $W = (V \cap W) \times (V^g \cap W)$, $g \in X \setminus N(V)$. Hence for each $\bar{x} \in \bar{P}$ of order 4 we have

$C_W(\bar{x}) = V \cap W = C_W(\bar{x}^2)$, since $\bar{x}^2 \in V$, a contradiction to [14, (1.1)]. Thus $\bar{P}$ is elementary and $\bar{X} \simeq L_2(q)$. Now $\bar{P} = \bar{V}$ by (2.4) and so $P = V$. But then V is weakly closed contrary to the assumption.

(3.3) *Case* (4) *of* (2.3) *cannot occur*.

Proof. Suppose false. Let $W_1 = WC_P(\bar{X})$, $A = V \cap W$, $L = VC_P(\bar{X})$ and $\tilde{X} = X/W_1$. Then by (2.5), W_1 is elementary, $\tilde{X} = \tilde{H}$, and $|\tilde{V}| = |\bar{V}| = q$.

By [12, (2.4)], $|A| = q^l$. If $|A| = q$ then, by (1.3), $|W| \leqslant q^n$, where $\tilde{X} \simeq U_n(q)$. But $|A^X| = |\tilde{H} : N_X(V)| > (q^n - 1)/(q - 1)$ by (2.4) and the structure of $U_n(q)$, a contradiction since A is a TI-set under the action of X. Hence $|A| \geqslant q^2$ and $|W| > q^n$.

Next we show

$$P \cap W = \langle A^y \mid V^y \leqslant C_X(V), y \in X \rangle. \tag{α}$$

Namely, let F denote the right side of Eq. (α). Then, since $C_{\bar{X}}(\bar{V}) = \langle \bar{V}^y \mid V^y \leqslant C_X(V) \rangle$ and $[W, V] = A$, it follows that $[W, C_{\bar{X}}(\bar{V})] = F$. Hence (2.5)(a) implies $W = F \times A^g$, since $F \times A^g$ is X-invariant. Thus $F = P \cap W = [P, W]$.

$$C_{\tilde{P}}(P \cap W) = \tilde{P}. \tag{β}$$

We have $\tilde{P} \cap Z(\tilde{X}) = 1$. Thus by the structure of $U_n(q)$, $C_{\tilde{X}}(\tilde{V})$ acts irreducibly on $\tilde{P}/\tilde{V}$. Let $\tilde{x} \in C_{\tilde{P}}(P \cap W) \setminus \tilde{V}$. Then by ($\alpha$), $\tilde{x}$ normalizes each $\tilde{V}^g \leqslant C_{\tilde{X}}(\tilde{V})$. Since by (2.3) and the structure of $U_n(q)$, $\tilde{V}^y \cap \tilde{P} = 1$ if $\tilde{V} \neq \tilde{V}^y$, it follows that $[\tilde{x}, \tilde{V}^y] = 1$. Hence $[\tilde{x}, C_{\tilde{X}}(\tilde{V})] = 1$, contradicting $C_{\tilde{X}}(\tilde{V})$ irreducible on $\tilde{P}/\tilde{V}$.

Now (β) implies $C_P(V(P \cap W)) = V(P \cap W_1) = C_P(L)$. Let $A \neq A^y \leqslant F$. Then $A^y = V^y \cap (W_1 \cap P)$. Thus, if $V^y \cap P > A^y$, then $V^y \cap P \neq 1$, a contradiction. Hence $A^y = V^y \cap P$ and is a TI-set in M.

Now, since V is a maximal Abelian normal subgroup of M, there is an $h \in M$ such that $L^h \neq L$. If for each $A^y \leqslant F$ we have $A^{yh} \leqslant L$, then $VF^h \leqslant L \cap L^h$. Hence $VF \leqslant L \cap L^{h^{-1}}$, contradicting $L = C_P(VF)$ by (β). Thus there is some $B = A^{yh} \nleqslant L$. If $B \cap L \neq 1$, then $B \leqslant C(VF)$, since B is a TI-set in M and $F = \langle A^y \mid V^y \leqslant C_X(V) \rangle$, a contradiction to ($\beta$). Thus $B \cap L = 1$. Especially there is some $A^y \leqslant F$ such that $[B, A^y] = A$. Hence $A \cup B^{A^y}$ is a partition of AB.

Let $R = C_M(A)$ and $\bar{R} = R/A$. We show that $\bar{B}$ is a TI-set in $\bar{R}$. By the above B is a TI-set in R. Let $g \in R$ such that $\bar{B} \cap \bar{B}^g \neq 1$. Then $AB \cap B^g \neq 1$. Hence $B^g = B^\alpha$ for some $\alpha \in A^d$ by the above. Thus $AB = AB^\alpha = AB^g$ and $\bar{B} = \bar{B}^g$.

Now by the action of $C_{\tilde{X}}(\tilde{V})$ on $\tilde{P}$ there is some $\bar{B}^r$, $r \in R$, such that $[\bar{B}, \bar{B}^r] \neq 1$. Since $P' \leqslant V$ we have $[\bar{B}, \bar{B}^r] \leqslant \bar{V}$. But by the above $|\bar{B}| = |A| > |\bar{V}|$ and so by the TI-set property of $\bar{B}$, $[\bar{B}, \bar{B}^r] = 1$, a contradiction. This proves (3.3).

(3.4) *Case* (3) *of* (2.3) *cannot occur.*

Proof. Suppose false. Then by [8], $\bar{X} \simeq \mathrm{Sp}(2n, q)$, since the perfect central extensions of Sp(4, 2) or Sp(6.2) are not generated by odd transpositions. By (1.4) and (2.5)(c), $W = \oplus W_i$, W_i are natural $\bar{X}$-modules. Let $A = W \cap V$, $I = WC_P(\bar{X})$. Then one shows as in (3.3) that there is some conjugate $A^g \leqslant P$, $g \in G$, such that $A^g \cap W = 1$.

Now $[W_1 \cap P, A^g] \leqslant A_1 = A \cap W_1$. Further, since $\bar{A}^g \cap \bar{V} = 1$, $[W_1 \cap P, A^g] \neq 1$. Now, as in (3.3), $A^g = V^g \cap P$ is a TI-set in M, whence $|A^g| \leqslant |A_1|$. But then $|A| = |A_1|$ and $W = W_1$ is the natural $\bar{X}$-module.

Let now $B = A^h$, $h \in X$, such that, by (2.5), $W = (W \cap P) \times B$. Then $I = (I \cap P) \times B$ and $B = V^h \cap M$, since $\langle V, V^h\rangle/AB \simeq L_2(q)$. Hence B is a TI-set in M.

Suppose $I > W$. Then $|I \cap P| > q^{2n-1}$ and $|I \cap P : C_{I\cap P}(\bar{A}^g)| = q$, since $[I \cap P, A^g] \leqslant A$. Now, since $C_{W\cap P}(\bar{V}\bar{A}^g)$ is a hyperplane of $W \cap P$ if one considers $W \cap P$ as a vector space over F_q with the partition $A^X \cap (W \cap P)$, it follows easily that $\bar{P} \cap \bar{X}$ is generated by $2n - 2$ conjugates of $\bar{V}\bar{A}^g$ in $C_{\bar{X}}(\bar{V})$, since $|\bar{P} \cap \bar{X}| = q^{2n-1}$. Hence, by the above, $|C_{I\cap P}(\bar{P} \cap \bar{X})| > q$. But by (2.5), $C_{I\cap P}(\bar{P} \cap \bar{X}) = C_{I\cap P}(P) = I \cap V = W \cap V$, a contradiction. Thus $W = I$ and $H = X$.

Let $\tilde{M} = M/P$. Next we show that

(α) $\tilde{B}$ is a TI-set in $\tilde{M}$.

Namely, assume that $\tilde{B} \cap \tilde{B}^g \neq 1$, $g \in M$. By the action of P on W we have $C_P(B) = P \cap W = C_P(b)$ for all $b \in B^{\#}$. Now $BP \cap B^g \neq 1$. So there is a $y \in (B^g)^{\#}$ such that $y = bp$, $b \in B^{\#}$, and $p \in P$. Suppose $o(p) = 4$. Then $p^b = p^{-1}$, since $(pb)^2 = y^2 = 1$. Hence $[p, b] = p^2 \in A$, since $A = \Phi(P)$. But since W is the natural $\bar{X}$-module, this implies that $p \in V(P \cap W)$, contradicting $o(p) = 4$. Hence $p^2 = 1$ and so $p \in C_P(b) = P \cap W$. Now again by the action of $\bar{X}$ on W each element of $W \setminus (W \cap P)$ is contained in some element of B^P. Hence $y \in B^r$, $r \in P$. But since B is a TI-set in M, $B^g = B^r$ and $BP = B^rP = B^gP$. This proves (α).

(β) Let $x \in B^{\#}$. Then $C_{\tilde{M}}(x) = \widetilde{C_M(x)}$. Further, $N_{\tilde{M}}(\mathrm{R}) = \widetilde{N_M(B)}$.

Let $\tilde{g} \in C_{\tilde{M}}(\tilde{x})$. Then $\tilde{B} = \tilde{B}^g$ by (α). As under (α), $B^g \leqslant (P \cap W)B$ and so $B^g = B^r$, $r \in P$. Hence $gr^{-1} \in N(B)$ and $x^{gr^{-1}} \in B \cap Px = x$. This implies that $\tilde{g} = \widetilde{gr^{-1}} \in \widetilde{C_M(x)}$. The proof of the second part is the same.

(γ) $Q = P \cdot C$, $[P, C] \leqslant A$. Further, $\Phi(C) = 1$, $[X, C] \leqslant W$, $[W, C] = 1$, and $|Q : P| \leqslant q$.

Since $N_X(V)$ contains an element acting regularly on $A^{\#}$ and on $(V/A)^{\#}$ it follows that $[A, Q] = 1$ and $[V, Q] \leqslant A$. Hence, by the three-subgroup lemma, $[B, Q] \leqslant C_Q(V) = P$. Therefore $Q \leqslant N(W)$, since $W = [P, B]B$ and by (β) $Q = P \cdot C_Q(B)$. Hence $C_Q(B) \leqslant N(X)$, since $X = \langle V^g \mid V^g \cap W \neq 1, g \in G\rangle$.

Now $C_X(V)$ acts irreducibly on $W \cap P/A$. Hence $[W \cap P, Q] \leqslant A$, since $[W \cap P, Q] < W \cap P$. The same argument implies that $[\bar{P} \cap \bar{X}, Q] \leqslant \bar{V}$. Hence by (1.2), Q induces inner automorphisms according to elements of $\bar{P} \cap \bar{X}$ on $\bar{X}$ and thus $Q = P \cdot E$, $E = C_Q(\bar{X})$.

Obviously $E \leqslant C_Q(W)$. By the three-subgroup lemma $E' = 1$, so $\mho^1(E) = \mho_1(EW)$ is normalized by X and thus $\Phi(E) = 1$. Let $W_1 = WE$. Then W_1 is an indecomposable $\bar{X}$-module, which is the extension of natural module W by a trivial. Thus by (1.5), $|W_1 : W| \leqslant q$ and $Q \leqslant PW_1$. If now $Q \cap W > P \cap W$, then by the action of X on W, $[P, Q] = P \cap W$. Hence $V(P \cap W) \lhd M$, a contradiction since V is a maximal Abelian normal subgroup. This shows $|Q : P| = |W_1 : W|$.

Now by (1.5), $W_1 = WC$, where $[P, C] \leqslant A$. Hence $C \leqslant F^*(M) = Q$, since C centralizes P/A and A. Since $|C| \geqslant |W_1 : W|$ it follows that $Q = PC$ which proves (γ).

(δ) $\tilde{Q} \cap \tilde{B} = 1$ and $\tilde{B}$ is strongly closed in $\tilde{Q}\tilde{B}$ with respect to $\tilde{M}$.

$\tilde{Q} \cap \tilde{B} = 1$ by what we have shown under (γ). Let $x \in (B^g)^{\#} \cap QB$, $g \in M$. Then $x = qb$, $q \in Q$, $b \in B^{\#}$. By (γ), $o(q) = 2$ or 4. Suppose first that $o(q) = 4$. Then as under (α), $q^b = q^{-1}$ and $q \in (Q \cap W_1)V$ by (γ), since $[q, b] \in A$. Hence $x \in (Q \cap W_1) VB = (Q \cap W_1) BV = W_1 V$. Thus $x = \alpha v$, $\alpha \in W_1^{\#}$, $v \in V \setminus A$, or $v = 1$. If $v \neq 1$, then $x \in C(v)$ since $\alpha^2 = 1 = v^2 = x^2$, a contradiction since $C_B(v) = 1$ for each $v \in V \setminus A$. Thus $x \in W_1$ and so $[B, B^g] = 1$. But this contradicts $o(q) = o(xb) = 4$.

Hence $o(q) = 2$ and $q \in C_Q(b) = Q \cap W_1$. Hence $B^g \cap W_1 \neq 1$ and $[B^g, W_1] = 1$, since $W_1 = (Q \cap W_1)B$ and $[Q \cap W_1, B^g] \leqslant Q \cap B^g = 1$.

Let $B^g = A^h$, $h \in G$. Then $\langle V, V^h\rangle/AA^h \simeq L_2(q)$ by [12, (2.4)] since $C_V(B^g) = A$. Let $L = WA^h$, $Y = \langle X, V^h\rangle$. Then $L \lhd Y$, since $[X, A^h] \leqslant W$. Further, since $O_2(Y) \leqslant N(V)$, $O_2(Y/L) \leqslant Z(Y/L)$. Hence (2.1) and (2.2) are satisfied for Y/L. But then by [1, 10], easily $(Y/L)/Z(Y/L) \simeq \mathrm{Sp}(2m, q)$ or $U_m(q)$. Since $PL/L \lhd C_Y(V)L/L$ the second case is impossible. In the first case, since $|L| = q^{2n+1}$ and $|A| = q$, it follows that $2m \leqslant 2n + 1$. But then $m = n$ and $Y = XL$, impossible since W is not normal in Y by $[A, V^h] = A^h$. This proves (δ).

Now, if $q = 2$, then either Q is extraspecial contradicting [6] or $Q = P$ is a direct product of an extraspecial with an elementary Abelian group. But then, since $V = Z(Q)$ is a TI-set and $A = Q'$ is not weakly closed in Q, the hypothesis of a recent theorem of Stroth [9] is satisfied. Hence, since $|Z(Q)| = 4$ and $|[Q/V, B]| > 2$, we get $M/Q \simeq O^-(6, 2)$ and $|Q| = 2^{10}$. But then by [16], $G \simeq M(22)$ and so V is not a TI-set.

So $q > 2$. Let $\bar{M} = M/Q$, $R = C_X(V)$. Then as shown $\bar{R} \simeq \mathrm{Sp}(2n - 2, q)$ and centralizes $\bar{B}$. Next we show:

(ϵ) $N_{\bar{M}}(\bar{R}) = N_{\bar{M}}(\bar{B})$ and $C_{\bar{M}}(\bar{R}) = \bar{B}\bar{K}$, $|\bar{K}|$ odd.

By (β) and (δ), $N_{\bar{M}}(\bar{B}) = \overline{N_M(B)}$ and so the maximal coimage F of $N_{\bar{M}}(\bar{B})$ normalizes $W = [\underset{\sim}{Q}, B]B$ by (γ). Hence $F \leqslant N(X)$ and so $\bar{F} \leqslant N_{\bar{M}}(\bar{R})$.

Let now E be the maximal coimage of $N_{\bar{M}}(\bar{R})$. Suppose there is an $e \in E$ such that $B^e \leqslant \underset{\sim}{Q}B$. Then $[P, B]/A$ and $[P, B^e]/A$ are both irreducible F_2R-modules. Hence either $[P, B] = [P, B^e]$ or $[P, B] \cap [P, B^e] = A$. In the first case, by the three-subgroup lemma, $[B, B^e] = 1$. Hence considering $L = WB^e$ and $Y = \langle X, V^h\rangle$, where $A^h = B^e$, one gets a contradiction as in (δ).

In the second case $P = [P, B][P, B^e]V$. Hence $[P, B^e]W/W$ is an R invariant complement to VW/W in PW/W, a contradiction since by the structure of $\mathrm{Sp}(2n, q)$ it follows that PW/W is an indecomposable F_2R-module. This proves the first part of (ϵ). So $C_{\bar{M}}(\bar{R}) \leqslant N_{\bar{M}}(\bar{B})$. Let $\bar{t} \in C_{\bar{M}}(\bar{R})$ such that $t^2 \in \underset{\sim}{Q}B$ for a coimage t of $\bar{t}$. Then by (β) and (γ), $t \in N(W)$, whence $t \in N(X)$. Further, by (1.5) we may assume that $[X, t] \leqslant W$, choosing the coimage appropriately. Thus $[W, t] = 1$ and so $t^2 = 1$, since $\langle t^2\rangle = \mho^1(W\langle t\rangle)$ is normalized by X. Hence $W\langle t\rangle$ is an indecomposable X-module and so by (1.5), $[P, wt] \leqslant A$ for some $w \in W$. But then $wt \in \underset{\sim}{Q}$ and so $t \in \underset{\sim}{Q}B$.

This shows $C_{\bar{M}}(\bar{R}) = \bar{B}\bar{K}$, $|\bar{K}|$ odd. By (α) and (δ), $\bar{B}$ is a TI-set in $\bar{M}$, whence $\bar{B}\bar{K}$ is by (ϵ) tightly embedded in $\bar{M}$. Further, since $|\bar{B}| > 2$, $O(\bar{M}) = 1$ by (ϵ). Thus $\bar{R}$ is a standard component of $\bar{M}$. Hence the main theorem of [5] implies either $\bar{R} \lhd \bar{M}$ or $\bar{R} \simeq L_2(4) \simeq \mathrm{Sp}(2, 4)$ and $\langle \bar{R}^{\bar{M}}\rangle \simeq HJ$, the Hall–Janko group. (Since there exists by the structure of X an element h acting regularly on $\bar{B}^{\#}$, the case $\bar{R} \simeq L_2(4)$ and $\langle \bar{R}^{\bar{M}}\rangle \simeq M_{12}$ does not occur.) If now $\bar{R} \lhd \bar{M}$ then, by (ϵ), $\bar{B} \lhd \bar{M}$ contradicting $\underset{\sim}{Q} = O_2(M)$. If $\bar{R} \simeq L_2(4)$ then $X/W \simeq \mathrm{Sp}(4, 4)$ and $|P/V| = 2^8$, $|V| = 16$. But by [2, (8.8)], the smallest F_2 representation of HJ is of dimension 12. This final contradiction proves (3.4).

Now (2.3) and (3.1)–(3.4) contradict each other, thus proving the main theorem.

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